AN IMPROVED PERFECTLY MATCHED LAYER FOR OPTICAL WAVEGUIDES

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1 Introduction

The beam propagation method (BPM) is widely used in numerical simulations of optical waveguides. A typical optical waveguide is an open waveguide with an unbounded transverse plane. Numerical computation can only be carried out in a finite computational domain and it is important to use accurate transverse boundary conditions or absorbing layers to simulate the radiation condition which allows only outgoing waves to propagate away from the axis of the waveguide [3]. The perfectly matched layer (PML) [1, 8] is a popular method for truncating an unbounded domain for wave propagation problems. Let x be an axis in the transverse plane of a waveguide, a PML corresponds to a complex change of x to $\hat{x} = x + i\int^x \sigma(\tau)d\tau$, for some real function σ. For weakly guiding optical waveguides, some propagating modes may decay rather slowly away from the axis, but the function σ has no damping effect on them. In this case, in order to accurately simulate the wave field, it is still necessary to use a relatively large transverse computational domain. In this paper, we introduce a real part in the definition of $\hat{x}$, i.e., $\hat{x} = x + \int^x [\xi(\tau) + i\sigma(\tau)]d\tau$. Based on the beam propagation method, we show that a suitable choice of ξ can significantly reduce the computational domain.

2 Basic equations

For a two dimensional waveguide and TE polarized light, the governing equation is the Helmholtz equation

$$u_{zz} + u_{xx} + k_0^2 n^2(x, z)u = 0, \tag{1}$$

where k_0 is a reference wavenumber, n is the refractive index, z is the axis of the waveguide. If the refractive index is a constant, say n_∞, for $|x| \geq H > 0$, we can define a continuous function σ, such that $\sigma(x) = 0$ for $|x| \leq H$ and $\sigma(x) < 0$ for $|x| > H$, and replace u_{xx} in (1) by $(1 + i\sigma)^{-1}\left[(1 + i\sigma)^{-1}u_x\right]_x$. This procedure corresponds to a change from $u(x, z)$ to $\hat{u}(\hat{x}, z)$ for $\hat{x} = x + i\int_0^x \sigma(\tau)d\tau$. The second order derivative u_{xx} is then changed to $\hat{u}_{\hat{x}\hat{x}}$. For the new equation, we can truncate the transverse variable at $x = \pm D$ for some $D > H$ and impose a boundary condition at $x = \pm D$. The simplest case is $u = 0$ at $x = \pm D$. For a plane wave that propagates toward $x = +\infty$, say $e^{-i\alpha x - i\beta z}$ with $\alpha > 0$ and $\alpha^2 + \beta^2 = k_0^2 n_\infty^2$, the boundary condition at $x = D$ gives rise to a reflected wave $Re^{i\alpha\hat{x} - i\beta z}$ with the reflection coefficient R satisfying $|R| = \exp\{2\alpha\int_0^D \sigma(\tau)d\tau\}$. On the other hand, for a propagating mode with a propagation constant β, say $u = e^{-\gamma(x-H) - i\beta z}$ for $x \geq H$, where $\gamma = \sqrt{\beta^2 - k_0 n_\infty^2} > 0$, the boundary condition at $x = D$ induces a reflected wave $Re^{\gamma(x-H) - i\beta z}$, with $|R| = e^{-2\gamma(D-H)}$. Clearly, the magnitude of R is related to the natural decay rate γ of the mode and it is not related to σ in the PML. If γ is small, the mode decays slowly in the cladding region and a large D is necessary.

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To damp the propagating modes, we introduce the complex variable $\hat{x} = x + \int_0^x [\xi(\tau) + i\sigma(\tau)]d\tau$, where ξ is a continuous function such that $\xi(x) = 0$ for $|x| \le H$ and $\xi(x) > 0$ for $|x| > H$. We then change the Helmholtz equation to

$$u_{zz} + (1 + \xi + i\sigma)^{-1}[(1 + \xi + i\sigma)^{-1}u_x]_x + k_0^2 n^2(x, z)u = 0. \tag{2}$$

Furthermore, we also truncate x and assume $u = 0$ at $x = \pm D$. In this case, the boundary condition at $x = D$ leads to a reflection coefficient with the same magnitude as before for the plane wave $e^{-i\alpha x - i\beta z}$. However, for the propagating mode $e^{-\gamma x - i\beta z}$, the reflection coefficient R now satisfies $|R| = e^{-2\gamma(D - H + \int_0^D \xi(\tau)d\tau)}$. This can be small if $\int_0^D \xi(\tau)d\tau$ is large.

Solving (2) is difficult for waveguide problems, since the solution is often needed over a large propagation distance, say from $z = 0$ to $z = L$. If the waveguide changes slowly with z and the wave field is dominated by its forward component, then the beam propagation method is useful and it solves the following one-way Helmholtz equation

$$u_z = -iB(z)u, \tag{3}$$

where the square root operator $B(z) = \sqrt{\partial_x^2 + k_0^2 n^2}$ for the original Helmholtz equation or

$$B(x) = \sqrt{(1 + \xi + i\sigma)^{-1}\partial_x[(1 + \xi + i\sigma)^{-1}\partial_x\ ] + k_0^2 n^2} \tag{4}$$

with our modified PML. From a consideration of energy conserving, Vassallo [7] derived the following more accurate one-way equation

$$u_z = \left(-iB - \sqrt{B}^{-1}\frac{d\sqrt{B}}{dz}\right)u. \tag{5}$$

It is equivalent to

$$\phi_z = -iB\phi, \quad \text{for} \quad \phi = \sqrt{B}u. \tag{6}$$

Another way to improve the accuracy of (3) is to use the single scatter approximation. In its conventional form, this approximation is formulated for a waveguide composed of piecewise z-invariant sections. It turns out [5] that in the continuous limit, the single scatter approximation solves the following equation

$$u_z = \left(-iB - \frac{1}{2}B^{-1}\frac{dB}{dz}\right)u. \tag{7}$$

The above equation can also be obtained through a wave field decomposition.

3 Discretization

Let z be discretized as $0 = z_0 < z_1 < z_2 < ... < z_m = L$, the one-way equations (3), (5) and (7) are used to find $u_j(x) \approx u(x, z_j)$, for $j = 1, 2, ..., m$, with a given u_0. For (3), a standard discretization is

$$u_{j+1} = e^{-ihB_{j+1/2}}u_j, \quad \text{for} \quad h = z_{j+1} - z_j \quad \text{and} \quad B_{j+1/2} = B(z_j + h/2). \tag{8}$$

With a reference refractive index n_*, we re-write B as $B = k_0 n_* \sqrt{I + X}$ for

$$X = (k_0 n_*)^{-2}\left\{(1 + \xi + i\sigma)^{-1}\partial_x[(1 + \xi + i\sigma)^{-1}\partial_x\ ] + k_0^2(n^2 - n_*^2)\right\}.$$

Various beam propagation methods can be obtained using different approximations of $\sqrt{I + X}$. A more efficient approach called split-step Padé method [2] is to use a rational approximation to the propagator directly:

$$e^{-is\sqrt{I+X}} \approx e^{-is}\prod_{j=1}^{p}\frac{I + c_j X}{I + b_j X} = e^{-is}\left(I + \sum_{j=1}^{p}\frac{a_j X}{I + b_j X}\right). \tag{9}$$

The above can be applied to (8), if we let $s = hk_0 n_*$ and replace X by $X_{j+1/2}$, where $X_{j+1/2}$ is related to $B_{j+1/2}$ through $B_{j+1/2} = k_0 n_* \sqrt{I + X_{j+1/2}}$. As in [9], the coefficients $\{a_j, b_j, c_j\}$ can be calculated in two steps. A $[p/p]$ complex coefficient rational approximation to $\sqrt{I+X}$ [4] is first inserted into $e^{-is\sqrt{I+X}}$, the result is then approximated by its $[p/p]$ Padé approximant. This two step approach utilizes the complex coefficient rational approximation to $\sqrt{I+X}$ which can effectively damp the evanescent modes.

Equation (5) can be similarly solved, using the equivalent formulation (6). To transfer from ϕ to u, a $[p/p]$ Padé approximation for $(I+X)^{-1/4}$ can be used. Finally, Equation (7) can be solved with an "alternative single scatter formulation" developed in [5]. This involves the standard propagator $e^{-is\sqrt{I+X}}$, the square root operator $\sqrt{I+X}$ and its inverse $(I+X)^{-1/2}$. For discretization, we use (9) and rational approximations to $(I+X)^{\pm 1/2}$.

4 Numerical Results

The numerical schemes described in the previous section have been implemented with a second order finite difference discretization for the transverse operator X. As an example, we consider a tapered waveguide studied in [6]. The parameters have been scaled and non-dimensionalized. This is a step index waveguide with $n = 1.1$ in the core and $n = 1$ in the cladding. The half width of the core, $d(z)$, satisfies

$$d(z) = 1, \quad \text{for} \quad z < 0, \quad \text{and} \quad d(z) = 0.5, \quad \text{for} \quad z > L = 100.$$

For $0 < z < L$, $d(z)$ is a linear function of z connecting $d = 1$ at $z = 0$ and $d = 0.5$ at $z = L$. The waveguide is symmetric in x, so that the core is defined as the region $|x| < d(z)$. For $z < 0$, we specify an incident wave which is identical to the symmetric propagating mode of the waveguide ($z < 0$). The reference wavenumber is $k_0 = 1$. The propagating mode decays to 0 as $|x|$ increases, but the decay rate is quite small. In [6], a computational interval of $|x| < 50$ is used in a BPM calculation. In this paper, we use a much smaller interval $|x| < 6$. In fact, we take $D = 6$ and $H = 5$. The PMLs are $5 < x < 6$ and $-6 < x < -5$. The functions ξ and σ are chosen to be

$$\xi(x) = -\sigma(x) = \begin{cases} 90(x-5)^3/(x-4) & \text{if } x > 5 \\ 0 & \text{if } -5 < x < 5 \\ 90(x+5)^3/(x+4) & \text{if } x < -5. \end{cases}$$

The x interval $[-6, 6]$ is then discretized with $\delta x = 0.025$ and the step size in z is $h = 1$. For the rational approximation (9), we choose $n_* = 1$ and $p = 6$. Numerical solutions for $|u(x, L)|^2$ are plotted in Figure 1. The solutions have a sharp decrease to 0 in the intervals $(-6, -5)$ and $(5, 6)$. This is caused by the non-physical PML. Three curves are shown in Figure 1. The solution from the one-way Helmholtz equation (3) is shown as the solid line. The dashed lines are the numerical solutions from the improved one-way equations (5) and (7). These two solutions are more accurate and there is no noticeable difference between the two.

5 Conclusion

A real part is introduced in the PML complex coordinate transform. It makes it possible to simulate wave propagation in a reduced computational domain for open optical waveguides, particularly when the propagating mode has a small decay rate in the cladding region. This modified PML is applied to the so-called split-step Padé implementation of the beam propagation method which approximates the propagator $\exp(-is\sqrt{I+X})$ directly by a rational function of X. These calculations are carried out for the one-way Helmholtz equation (3) and its two improvements — the energy-conserving equation (5) and the single scatter approximation (7). Numerical results are obtained for a tapered waveguide using a computational interval that is much smaller than the one used in a previous calculation.

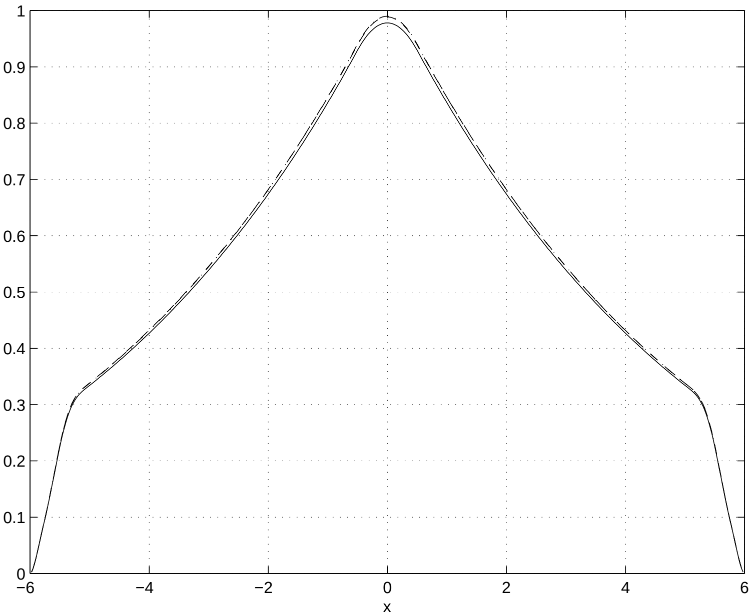

Figure 1: The solid line is for $|u(x, L)|^2$ calculated from (3). The dashed lines are $|u(x, L)|^2$ calculated from (5) and (7).

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