AN EFFICIENT NUMERICAL METHOD FOR DFB STRUCTURES BASED ON PERIOD DOUBLING AND RATIONAL APPROXIMATIONS

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## 1 Introduction

For many integrated photonic devices, it is important to accurately calculate both the transmitted and reflected wave field. This is certainly so for distributed-feedback lasers, since they incorporate reflections from a grating distributed throughout the active region. For these problems, bidirectional propagation methods are required. As an alternative to iterative bidirectional propagation methods which propagate the wave field back and forth many times, non-iterative methods [1] have been developed based on operators and their rational approximations.

Consider a planar waveguide with the main propagation direction in $z$, and assume that the waveguide is $z$-invariant for $z < 0$ and $z > L$. In the interval $[0, L]$, the waveguide is composed of $m$ distinct $z$-invariant sections $z_{j-1} < z < z_j$ for $j = 1, 2, ..., m$, where $z_0 = 0$ and $z_m = L$. For TE polarized light, the governing equation is

$$u_{zz} + u_{xx} + k_0^2 n^2(x, z)u = 0,$$

where the refractive index $n$ is piecewise constant in $z$. That is, $n(x, z) = n_j(x)$ for $z_{j-1} < z < z_j$ and $j = 0, 1, 2, ..., m, m+1$. Here we further assume that $z_{-1} = -\infty$ and $z_{m+1} = +\infty$. Non-iterative propagation methods can be developed based on the following square root operators

$$L_j = \sqrt{\partial_x^2 + k_0^2 n_j^2(x)} \quad \text{for} \quad j = 0, 1, 2, ..., m+1,$$

and the forward and backward propagation operators $P_j^{\pm} = \exp(\mp i(z_j - z_{j-1})L_j)$, for $j = 1, 2, ..., m$. Rational approximations to the square root operator $L_j$ can be obtained based on $L_j = k_0 n_* \sqrt{I + X_j}$ for $X_j = [\partial_x^2 + k_0^2 n_j^2(x) - k_0^2 n_*^2]/(k_0^2 n_*^2)$, where $I$ is the identity operator and $n_*$ is a reference refractive index. The standard Padé approximation gives

$$L_j \approx k_0 n_* \left( I + \sum_{k=1}^{p} \frac{a_k^{(p)} X_j}{I + b_k^{(p)} X_j} \right) = k_0 n_* \prod_{k=1}^{p} \frac{I + c_k^{(p)} X_j}{I + b_k^{(p)} X_j}$$

where $a_k^{(p)}$, $b_k^{(p)}$ and $c_k^{(p)}$ are real constants. However, the evanescent modes (corresponding to eigenvalues of $X_j$ which are less than $-1$) are not correctly modeled. This has motivated the complex coefficient rational approximations to the square root operator, such as the rotated branch-cut method in [2] and the modified Padé method in [3].

The method developed in [1] calculates a $2 \times 2$ operator matrix $G$ which maps the forward and backward components of $u$ at $z = 0-$ to the forward and backward components of $u$ at $z = L+$. The matrix $G$ involves the operators $P_j^{\pm}$ and $L_j^{-1} L_{j+1}$. Complex coefficient rational approximations are used in $L_j^{-1} L_{j+1}$, but not

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in $P_j^{\pm}$. The choice of real Padé coefficients for $L_j$ in $P_j^{\pm}$ is essential for the stability of their method. If the exact operators $P_j^{\pm}$ are used, their method is actually unstable. Thus, in their method, the evanescent modes are incorrectly treated as propagating modes in each $z$-invariant sections.

In this paper, we develop a non-iterative bidirectional propagation method based on the reflection and transmission operators. Our method is truly stable and complex coefficient rational approximations are used in the square root operators for both the transition and propagation steps. Our method also has an important property for periodic waveguides. If the interval $[0, L]$ involves $M$ periods, then only $\log_2 M$ steps are needed to complete the calculation.

## 2 A Stable Bidirectional Propagation Method

We first consider the case that the wave field is generated by an incident wave coming from $z = -\infty$ and assume that there is only an outgoing wave for $z > L$. In each $z$-invariant section, we can decompose the wave field into forward (to $z = +\infty$) and backward (to $z = -\infty$) components, and they satisfy one-way Helmholtz equations involving the square root operator. For $z_{j-1} < z < z_j$, we have $u = u^+ + u^-$ and

$$u_z^+ = -iL_j u^+, \quad u_z^- = iL_j u^-.$$

The reflection operator at $z \in (z_{j-1}, z_j)$ maps the forward component to the backward component, i.e., $R(z)u^+(\cdot, z) = u^-(\cdot, z)$. The transmission operator maps the forward component at $z$ to the forward component at $z = L+$, i.e., $T(z)u^+(\cdot, z) = u^+(\cdot, L+)$. Taking the limit towards the discontinuities, we have $R(z_j-)$, $T(z_j-)$, $R(z_j+)$ and $T(z_j+)$, for $j = 0, 1, 2, ..., m$.

Our problem is to calculate $R(0-)$ and $T(0-)$ from the given $R(L+) = 0$ and $T(L+) = I$, since only outgoing waves are allowed for $z > L$. We need a "transition" formula that leads the operators from $z_j+$ to $z_j-$, and a "propagation" formula that moves the operators from $z_j-$ to $z_{j-1}+$. These formulas are well known and are given in the following algorithm:

For $j = m, m-1, ..., 0$,
    $C = L_j^{-1} L_{j+1} [I - R(z_j+)][I + R(z_j+)]^{-1}$
    $R(z_j-) = (I + C)^{-1}(I - C)$
    $T(z_j-) = T(z_j+)[I + R(z_j+)]^{-1}[I + R(z_j-)]$
    If $j > 0$, then
        $R(z_{j-1}+) = P_j^+ R(z_j-) P_j^+$
        $T(z_{j-1}+) = T(z_j-) P_j^+$
    end
end

The operator $P_j^+$ has already been defined in Section 1.

This gives rise to a bidirectional propagation method. The operators $R(0-)$ and $T(0-)$ give the reflected wave and the transmitted wave, respectively. The algorithm is non-iterative, since only one sweep in $z$ (from $z = L+$ to $z = 0-$) is needed. With further rational approximations to the square root operator and $P_j^+$, this approach can become useful for practical calculations. However, this is a sequential algorithm and $m$ steps are required, since we have to jump over each discontinuity and move through each $z$-invariant section in a well defined sequence. In the next section, we describe a more efficient algorithm for periodic waveguides.

## 3 Period Doubling

In this section, we consider a waveguide with a periodic grating. We assume that there are only two distinct refractive index profiles, $n_0(x)$ and $n_1(x)$. The discontinuities are located at

$$z_0 = 0, \; z_1 = l_0, \; z_2 = z_1 + l_1, \quad z_3 = z_2 + l_0, \quad z_4 = z_3 + l_1, \quad ...., \quad z_{m-1} = z_{m-2} + l_0,$$

where $m = 2M$ is an even integer. The refractive index profile in the interval $(z_{j-1}, z_j)$ is $n_j(x)$ as before, but

$$n_j(x) = \begin{cases} n_1(x) & \text{if } j \text{ is odd,} \\ n_0(x) & \text{if } j \text{ is even.} \end{cases}$$

From $z = 0$ to $z = z_{m-2} = (M-1)(l_0+l_1)$, we have $M-1$ periods with the period $l = l_0+l_1$. For $z > z_{m-1}$, the waveguide is $z$-invariant with $n(x,z) = n_0(x)$, but we add an additional point $z_m = z_{m-1}+l_1 = Ml = L$, such that there are a total of $M$ periods from $z = 0$ to $z = z_m$. However, $z_m$ is not a point of discontinuity of the refractive index.

For this problem, we have a pair of operators $R(0-)$, $T(0-)$ as in Section 3, for the scattering problem related to incident waves coming from $z = -\infty$. These two operators will now be denoted as $R_M^-$ and $T_M^-$, although the transmission operator maps the incident wave at $z = 0-$ to the outgoing wave at $z = L-$ (not $z = L+$). These operators satisfy

$$R_M^- u^+(\cdot, 0-) = u^-(\cdot, 0-), \quad T_M^- u^+(\cdot, 0-) = u^+(\cdot, L-),$$

where $L = z_m$. Similarly, for incident waves coming from $z = +\infty$, we have a pair of operators such that

$$R_M^+ u^-(\cdot, L-) = u^+(\cdot, L-), \quad T_M^+ u^-(\cdot, L-) = u^-(\cdot, 0-).$$

It has been established [4, 5] that these four operators for a waveguide with $2s$ periods can be easily deduced from the corresponding four operators with $s$ periods, where $s$ is an integer. We have

$$\begin{aligned} R_{2s}^- &= R_s^- + T_s^+(I - R_s^- R_s^+)^{-1} R_s^- T_s^- \\ T_{2s}^- &= T_s^-(I - R_s^+ R_s^-)^{-1} T_s^- \\ R_{2s}^+ &= R_s^+ + T_s^-(I - R_s^+ R_s^-)^{-1} R_s^+ T_s^+ \\ T_{2s}^+ &= T_s^+(I - R_s^- R_s^+)^{-1} T_s^+. \end{aligned}$$

Therefore, when $M$ is an integer power of 2, we first calculate the four operators $R_1^\pm$ and $T_1^\pm$ based on the procedures outlined in Section 2, then we use $\log_2 M$ period doubling steps (for $s = 1, 2, 4, ..., M/2$) based on the above formulas. Compared with the sequential algorithm in Section 3, this method can significantly speed up the calculation.

## 4 Numerical Implementation and Examples

When the transverse variable $x$ is discretized, say by $N$ points, the operator $\partial_x^2 + k_0^2 n^2$ is approximated by a matrix, then all the reflection and transmission operators are reduced to $N \times N$ matrices. In the following example, we chose $l_0 = 0.0621$ $\mu$m, $l_1 = 0.06125$ $\mu$m. The refractive index takes the constant $n_{co} = 3.3$ in the core and $n_{cl} = 3.2$ in the cladding. The profiles $n_0$ and $n_1$ correspond to sections with core width 0.5 $\mu$m and 0.1875 $\mu$m, respectively. Our problem is to calculate the transmitted and reflected waves for a given incident wave which is chosen to be the fundamental (symmetric) propagating mode of the waveguide (for $z < 0$ with a core width of 0.5 $\mu$m).

With a Perfectly Matched Layer[6], we discretize $x$ from $-1$ $\mu$m to 1 $\mu$m with 512 points and a second order finite difference method is used to approximate the transverse operator. Complex coefficient rational approximations developed in [3] are used with $p = 6$, and the operator $P_j^+$ is further approximated with a [2/2] Padé approximation to the exponential function. In Figure 1, the reflectivity and the total power are shown with different wavelength of the incident wave. Our results are consistent with the calculation in [1]. These results are also compared with an exact method where the operators $L_j$ and $P_j$ are exactly evaluated. No noticeable difference has been observed.

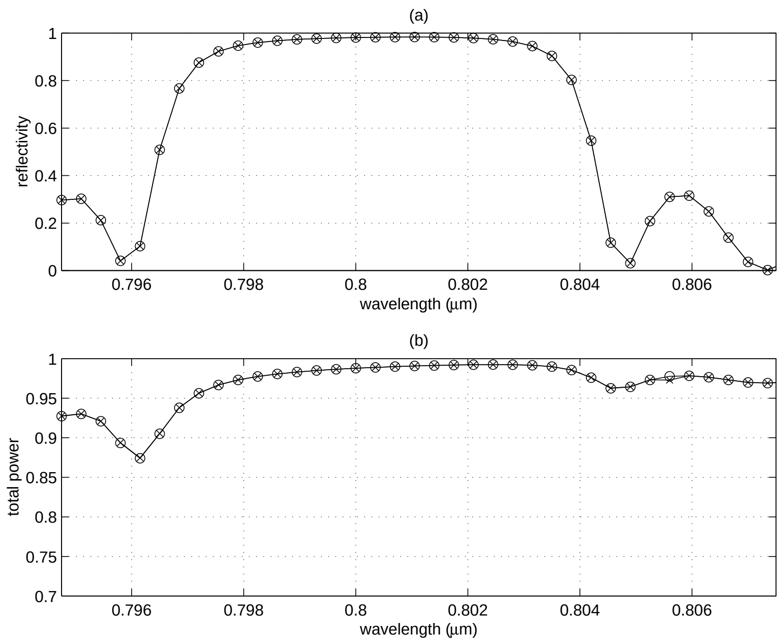


Figure 1: Reflectivity (a) and total power (b) for a DFB structure with 256 periods.

## 5 Conclusion

A non-iterative bidirectional propagation method has been developed for waveguide problems using the reflection and transmission operators. The method is stable, even when complex coefficient rational approximations are used to approximate the square root operator (so as the evanescent modes are damped). The method is further enhanced with a period doubling process for periodic waveguides, such as a DFB structure. Numerical examples indicate that the method is highly accurate and quite efficient.

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